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Hugh MacColl after One Hundred Years

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On the growth and use of a symbolical language

Hugh MacColl

EDITOR'S NOTE

[Editors' note: This paper was communicated by the Rev. Robert Harley, F. R. S., and read at the Ordinary meeting of the *Manchester Literary and Philosophical* Society on 22 March 1881. The full paper, signed "Hugh M'Coll, Esq., BA.", appeared first (without the abstract) in the *Memoirs* of the Society, series 3, 7, 1882, 225--248. The (English) abstract below appeared first in the *Proceedings* of the Society, 20 (8), session 1880-1881, 103. The French abstract below has been provided by the editors.]

1 In an article on "Symbolical Reasoning", in a recent number of 'Mind' (n° 17, Jan. 1880), I have described the relation between symbolical reasoning and ordinary verbal reasoning as analogous to that between machine labour and ordinary manual labour. To trace this analogy through all its various points of resemblance would take too long; but there is one point which deserves some notice, as it bears more especially upon the present subject.

2 For what kinds of operations are machines usually invented? A little reflection will show that one common and prominent characteristic of such operations is *sameness; we employ machines to perform operations which have to be frequently repeated, and repeated in the same unvarying manner.* Sewing-machines, knitting-machines, reaping-machines, and, in fact, the great generality of machines, however widely they may differ in other respects, resemble each other in this.

3 For what kinds of expressions and relations, mathematical or logical, do we usually invent symbols? We shall find, as before, that the common characteristic of such expressions and relations is *sameness*—that they are expressions and relations *which have to be repeated frequently.* When any complex expression or relation is perceived to have a tendency to

recur again and again, we economize thought, time, and space if we denote this expression or relation by some simple, suggestive, and easily formed symbol which we may always recognize as doing duty for its more complex equivalent.

4 The representive symbols thus invented combine afterwards among themselves into new expressions and relations of more or less complexity, and give birth, in their turn, when the necessity or convenience arises, to fresh representative symbols, whose abbreviating power bears, on an average, the same ratio to that of the symbols they displace, as the abbreviating power of the latter bears to that of *their* immediate progenitors. In strict conformity with this law of symbolical growth the science of mathematics has gradually attained its present wonderful power within the limits of its application; and in strict conformity with the same law, the science of logic, which is now evidently entering on quite a new phase of existence, will probably before long, and within much wider limits of application, surpass the achievements of mathematical science itself.

5 Now it is clear that the power and progress of any symbolical language must depend very largely upon the judgment exercised, first, as to whether, in any proposed case, a new symbol is really required, or would on the whole be useful, and, secondly (supposing the need of a new symbol to be admitted), as to the kind of symbol that should be selected. With regard to the first point, we must remember that the introduction of a fresh symbol is always accompanied by the disadvantage that it adds a fresh item to the load which the memory has to carry, and it is only when its advantages more than outweigh this very serious drawback that it should be admitted as a permanent addition to the existing vocabulary. Can we discover any general principles or rules which should guide us in this important matter of admission or rejection? Let us examine a few of the symbols which we now possess, and see whether any such rules can be discovered.

6 The ratio which the circumference of a circle has to its diameter, namely, 3.14159 &c., is one that occurs frequently, and for this reason mathematicians express it by a single arbitrary symbol π. The ratio which the diagonal of a square has to its side, namely, 1.41421 &c., is another ratio which also occurs frequently, and yet mathematicians do *not* express this by any single arbitrary symbol, nor would any mathematician think the introduction of such a symbol desirable. Why is this? The answer is obvious: the latter ratio may be expressed, *without any fresh definition or explanation,* by a very brief and simple combination of existing symbols, namely by the combination $\sqrt{2}$; while we know of no brief and easily formed combination of existing symbols, *requiring no fresh definition,* which would accurately and unambiguously express the former ratio.

7 From these and other analogous examples we may safely assume as one guiding principle, that some conventional symbol of abbreviation should be used as a substitute for any expression that has a tendency to recur frequently, *provided that no suitable combination of existing symbols* (i.e. a combination short, simple, and requiring no fresh definition or explanation) *can be found to replace it.*

8 The next point is, as a rule, more important and also less easily decided. It is this: — Granting the necessity for some new symbol of abbreviation, what kind of symbol should be selected?

9 In the ease of the symbol π, to which we have already alluded, this question of suitable selection is, it is true, of secondary importance; almost any arbitrary symbol of easy formation would have done just as well; but this is an exception to the general rule. Consider the symbol a^n, which has been invented as an abbreviation for the product of n

equal factors, each equal to *a*; that is, a^2 for *aa*, a^3 for *aaa*, and so on. If the first of these products, namely *aa*, were the only one that had a tendency to recur, we may be quite sure that mathematicians would remain satisfied with it in its original form, and would never have accepted the innovation a^2 as its equivalent. But since *aaa*, *aaaa*, &c., have also the same tendency of frequent recurrence, the appropriateness of the symbol selected is evident: the numerical index reminds us of the number of equal factors; and we are at once provided with a more effective notation for considering the properties and relations of all expressions that are products of equal factors, as, for instance, in the binomial theorem.

10 Let us now examine the *raison d'être* of that remarkable class of symbols which were invented at a more advanced stage of the science (by whom I know not), and which give such a wonderful sweep and power to symbolical language generally, logical as well as mathematical; I refer to that class of symbols of which $f(x)$ may be taken as a specimen. This symbol denotes *any complex expression whatever* (mathematical or logical) that contains the simpler expression *x*, *in any relation whatever* as one of its constituents. What was the special need which this symbol was invented to supply?

We have often to consider what an expression would become if one of its constituents were taken away and a fresh constituent put into its place, just as people sometimes speculate as to what would be the effect upon a ministerial policy if a certain member of the cabinet were to resign and a certain other person appointed in his place. If $f(x)$ denote the expression of which *x* is a constituent, then $f(a)$ will denote the new expression which is formed by substituting *a* for *x*. To take a simple case, let $f(x)$ denote the algebraical expression

$$\frac{6}{3-x} + x + 5;$$

then $f(2)$ will denote

$$\frac{6}{3-2} + 2 + 5,$$

and will be equal to 13; $f(0)$ will denote

$$\frac{6}{3-0} + 0 + 5,$$

and will be equal to 7; $f(x-5)$ will denote

$$\frac{6}{3-(x-5)} + (x-5) + 5,$$

and will be equal to

$$\frac{6}{8-x} + x; \text{ and so on.}$$

11 The last symbol $f(x-5)$ warns us of a danger to be carefully guarded against in the introduction of fresh symbols, namely *the danger of ambiguity*. The meaning here attached to it *might* in certain cases be confounded with an older and commoner meaning; for the symbol $f(x-5)$ also denotes *the product of the two factors f and x* – 5. How is this danger of ambiguity to be guarded against? We might, it is true, guard against it by adopting, instead of *f*, a totally new symbol of some unwonted shape; but this is a course to be avoided if possible. Strange-looking symbols somehow offend the eye; and we do not take to them kindly, even when they are of simple and easy formation. Provided we can avoid ambiguity, it is generally better to intrust an old symbol with new duties than to employ

the services of a perfect stranger. In the case just considered, and in many analogous cases, the context will be quite sufficient to prevent us from confounding one meaning with another, just as in ordinary discourse we run no risk of confounding the meanings of the word *air* in the two statements—"He assumed an air of authority," and "He resolved the air into its component gases." In the special case of $f(x - 5)$ no ambiguity exists when the letter f is used in no other sense throughout the investigation on which we happen to be engaged; and when it *is* used in another sense, all risk of confusion is obviated by simply employing instead of f, in the expressions $f(x)$, $f(x - 5)$, &c., some other symbolic letter, such as ϕ.

12 We may recapitulate the results so far arrived at thus: —

1. A new symbol, or an old symbol with a new meaning, to be accepted as a permanent addition to the existing stock, should represent all expression or relation that tends to occur frequently, and that cannot be logically expressed by any short combination of existing symbols.
2. Provided we can avoid ambiguity, it is better to employ an old symbol in a new sense than to invent a totally new symbol.
3. The symbol chosen should be of short and easy formation and lead to symbolic expressions of simple and symmetrical forms.

13 The illustrations hitherto given have been borrowed from mathematics, because these are more familiar to the general reader than any that could have been taken from symbolical logic; yet the latter is in every respect the simpler science of the two,—simpler in the conceptions represented by symbols, simpler in the smallness of the number of symbols employed, and simpler in the mechanical operations that have to be performed. In claiming this advantage for logic over mathematics, I speak solely of that scheme of symbolical logic which I, rightly or wrongly, consider the simplest and most effective, namely the scheme which I have explained and illustrated in 'Mind,' in the 'Proceedings' of the London Mathematical Society, in the 'Educational Times,' and in the 'Philosophical Magazine'. According to this scheme the whole and sole duty of the logician is to investigate the relations in which *statements* (i.e. assertions and denials) stand towards each other. For all practical reasoning-purposes a *statement* may be defined as anything that conveys directly through a bodily sense (as the eye or ear) any information (true or false) to the mind. In this sense a nod or a shake of the head is a perfectly intelligible statement. The Union-Jack fluttering from the mast of a ship conveys as clear and definite information as the words "This is a British ship" shouted through the captain's speaking-trumpet; and therefore the flag is as much a statement in the logical sense as the words; and, like the words, it may (as we know by experience) be a true or a false statement.

14 Logic, then, being concerned with statements, the analogy of ordinary algebra suggests the propriety of denoting simple statements by single letters, and the relations in which statements stand to each other either by the relative positions of the statement-letters, or by separate and distinct symbols. Therefore a very important inquiry in laying the foundation of a practical symbolical calculus for solving logical problems is this: —What are the characteristics and relations which most frequently distinguish or connect the statements of an argument? Foremost among distinctions we shall find that of *truth* and *falsehood.* All intelligible statements may be divided into two great classes, the *true* and the *false.* Every statement must belong to one or other of these two classes, *though we may not always know to which.* If it were not for this element of uncertainty, reasoning would be purposeless, and logic would have no *raison d'être.* This uncertainty (sometimes real and

sometimes only hypothetical) suggests the convenience of dividing the statements of any argument upon which we happen to be engaged into three distinct classes, the *admittedly true,* the *admittedly false,* and the *doubtful.* Borrowing a hint from mathematical probability, we may denote any statement belonging to the first class by the symbol 1, and any statement belonging to the second class by 0, while any doubtful statement (whether the doubt be real or hypothetical) may be denoted by any symbol we choose except these. Now, generally speaking, in the course of any consistent argument or investigation the boundaries of these three classes will be found to be gradually changing; the first two classes, the *admittedly true* and the *admittedly false,* though never encroaching upon each other's ground, will both constantly encroach upon the ground of the third, the *doubtful.*

15 Statements may also, independently of their truth or falsehood, be divided into two distinct classes, namely *assertions* and *denials.* Every assertion either claims or has already obtained admission into the class denoted by the symbol ; while its denial contests its right to this symbol, which it claims for itself, and seeks to brand it, as an impostor, with the symbol 0. As long as these two claimants belong to the class of doubtful statements, all that we can say about them is, that the one (either the assertion or its denial) must be true, and the other false.

16 The denial of any assertion may be conveniently denoted by an accent, thus: —Let x denote the statement, "He is in England;" then x' will denote "He is *not* in England."

17 The statements hitherto spoken of are simple or elementary statements—that is, statements represented each by a single letter, or a single letter and an accent. Any statement that requires more than one letter to express it may be called a *complex* statement. The principal relations by virtue of which simple statements combine into complex ones are three—namely, *conjunction, disjunction,* and *implication,* corresponding respectively to the three conjunctions *and, or, if.* The first relation is generally symbolized (like multiplication in ordinary algebra) by simple juxtaposition, and occasionally, though never necessarily, by the symbol ×; the second (like addition in ordinary algebra) by the symbol +; and the third by the symbol :, as in the following examples: —

18 Let x denote the statement "He will go to Paris;" and let y denote the statement "I shall go to York." Then $x\,y$ denotes the compound statement "He will go to Paris *and* I shall go to York;" the symbol $x + y$ denotes the disjunctive statement "He will go to Paris *or* I shall go to York;" and $x : y$denotes the implication "*If* he goes to Paris I shall go to York."

19 A compound statement, as $a\,b\,c$, claims the symbol 1 for *every one* of its factors; a disjunctive statement, as $a + b + c$, claims the symbol 1 for *one at least* of its terms; and the implicational statement $a : b$ claims the symbol 1 for the consequent b, *provided the antecedent a is entitled to it,* but neither claims nor disclaims it for b if a is *not* entitled to it.

20 Brackets are used when necessary to collect the different elements of a complex statement, and so prevent any uncertainty respecting to what complex statement any element or relational symbol belongs. Thus, the compound statement $a(b + c)$, formed by the two *factorsa* and $b + c$, is a very different statement from the disjunctive statement $ab + c$, formed by the two *termsab* and c. So, again, is $(a : b + c)'$, the denial of the whole implicational complex statement $a : b + c$, a very different statement from $a : b + c'$, in which the symbol of denial affects only the element .

21 Reciprocal implications—that is, compound implications of the form $(a : b)(b : c)$—occur so frequently that a symbol of abbreviation is convenient. Borrowing again from the

existent mathematical stock, we may use[†]either ∷ or =. Thus either the symbol $a ∷ b$ or the symbol $a = b$ may be taken as an abbreviation for the reciprocal implication $(a : b)(b : a)$.

22 The symbol $f(x)$ has been already considered, and is employed in logic in the same sense as in mathematics; that is to say, it denotes any statement whatever that contains x as one of its constituents; but the symbol $f'(x)$, for which no logical meaning analogous to its mathematical one is likely to turn up, may be conveniently employed as an abbreviation for $\{f(x)\}'$.

23 These are the only symbols that need be employed in the system of symbolical logic which I advocate, and they are amply sufficient not only for the complete solution of any logical problem that I have ever seen solved by any other method, but also for the complete solution of many problems which, I think, it would be difficult to solve by any other method with which I am at present acquainted.

24 The rules which I have proposed for observance in introducing new symbols are, I believe, sound, and I have followed them myself to to the best of my ability. As the science advances, other symbols will, no doubt, become necessary; but they should be introduced slowly, and not till their utility is made clearly manifest.

25 My statement in 'Mind,' that though $a : b$ implies $a' + b$, it is not equivalent to it, has been called in question, my critics maintaining that there is no real difference between the conditional statement "If a is true, b is true," and the disjunctive statement "Either a is false or b is true." Now, I admit at once that, in the ordinary language of life, disjunctive statements are often made which convey, and are intended to convey, a *conditional meaning,* and, further, that the example which I gave in illustration, namely "He will either discontinue his extravagance, or he will be ruined," is one of them. Many statements, however, are made in common life which are tacitly understood to convey a stronger meaning than logically and literally belongs to them. Take, for instance, the well-known expression, "He will never set the Thames on fire." In its literal sense this very harmless-sounding statement does not commit one to much; it may, with equal safety, be applied to the cleverest man living and to the most incapable idiot. What the practical reasoner would be concerned with in making use of any evidence conveyed to him in such terms would be the *intended* meaning of the speaker; and if his argument should be of such a nature as to necessitate the employment of symbols, the symbol for the statement should denote its *intended* and not its literal meaning. The real question in dispute is this, does the conditional statement "If a is true b is true," *as I define and symbolize it,* convey a meaning in any way different from the disjunctive statement "Either a is false or b is true," *as I define and symbolize it?*

26 My argument in 'Mind' was, that since the *denial* of the first, namely " a may be true without b being so," conveys less information than " a is true and b is false," which is the denial of the second, the conditional disjunctive statements of which these are the respective denials cannot be equivalent. As the non-equivalence of the denials, however, is much more evident than that of the affirmative statements, it will be well worth while to give, if possible, a more direct proof of the non-equivalence of the latter.

27 As it can easily be shown that $a : b$ is equivalent to $a' + b$, the question may be narrowed to this, is the implication $1 : \alpha$, in which α denotes $a' + b$, equivalent to the simple affirmation α? It seems to me that $1 : \alpha$ and α differ in pretty much the same way as the statement "It is well-known that tin is heavier than zinc," and the simpler affirmation "Tin is heavier than zinc;" that is to say, the former implies the latter, but is not implied in it. The

statement $1 : \alpha$, in addition to claiming the symbol 1 for itself, asserts that its *protégéα* has fairly made good its right to it; whereas α only claims this symbol on its own account. The symbol $(1 : \alpha)'$, which is the denial of $1 : \alpha$, may be read, " α is not necessarily true;" whereas α', the denial of α, is much stronger, and asserts positively that α is false. It follows from the law of logic called *contraposition* that the denial of the weaker (or implied) statement is stronger than and implies the denial of the stronger (or implying) statement.

28 The disjunctive statements of ordinary language may be divided into *conditional* disjunctives and *unconditional* (or pure) disjunctives—the former being those already referred to as conditional in meaning though disjunctive in form. Among these may be classed the famous disjunctive of Edward I., "By God, sir Earl, you shall either go or hang" (symbolically, $g + h$), the meaning of which is evidently, "If you don't go, I will have you hanged." The king, by a not uncommon trick of speech, used the weaker statement in order to express a stronger meaning. The earl, in his reply, "By God, sir king, I shall neither go nor hang" ($g'h'$), secures emphasis of a more direct kind by flatly denying even the king's weak disjunctive, instead of the much stronger conditional statement which would more logically, though less emphatically, express the king's real meaning. The denial of "If you won't go, I will have you hanged," would be the very mild assertion, "I may refuse to go without your having me hanged," a mode of speech which would not at all have suited the temper of Earl Bigod.

29 As an instance of a simple unconditional disjunctive, we may take the statement, "We shall either go to Brighton or Hastings this summer." There appears to me to be a fundamental difference between the class of disjunctives of which this is a type, and the conditional class of disjunctives previously illustrated. At the same time it must be admitted that in common language, just as statements which are conditional in meaning are often expressed in a disjunctive form, so real disjunctives unconditional in meaning are often expressed in a conditional form. The last example, for instance, "We shall either go to Brighton or to Hastings this summer," might, according to usage and without any perceptible difference of meaning, be expressed as "If we don't go to Brighton this summer, we shall go to Hastings," or in the same words with Brighton and Hastings interchanged. The fact, however, that conditionals and disjunctives are frequently confounded in ordinary untechnical language, is no reason why they should be so in formal or symbolical logic. Even if I have not succeeded in satisfactorily proving that $a : b$ and $a' + b$ are not synonymous, it is safest, I think, to adopt my view in actual practice. Let it be observed that the hypothesis of non-equivalence commits one to less, and therefore involves less risk to the inferred conclusions. My critics admit with me that $(a : b) : (a' + b)$ is a correct formula; but they would also add the formula $a' + b : (a : b)$, the validity of which I deny. If I am wrong, I am open to the charge of seeking to deprive logic of a new formula which might possibly prove useful, but whose utility has yet to be proved. If my opponents are wrong, they are open to the graver charge of seeking to introduce an erroneous formula, which not only can render no service in reasoning, but might even seriouslyendanger our conclusions.

30 As this article is an attempt to explain and illustrate the laws which necessitate the growth, and the principles which determine the form, of a symbolical language, I hope it will not be considered either irrelevant or egotistical, if I give a brief account of the development of my own method. By "my own method" I mean simply the *method which I discovered* (including those features which it has in common with the prior methods of

others), as well as those characteristics which are peculiar to itself. What these are I leave to others to decide. The question is certainly irrelevant to the expressed object of this article; and its discussion would only provoke the natural impatience of the reader. I only mention this at all in order to explain that I use the possessive pronoun *my* merely as a convenient abbreviation, and in a sense which cannot possibly give offence to any of my fellow workers.

31 As I stated in my third paper in the 'Proceedings' of the London Mathematical Society, my method originated in a question in probability proposed in the 'Educational Times' for June 1871[‡]. The question (as I understood it) may be thus generalized : —"Given that the variables x, y, z are each taken at random between given limits, what is the chance that an assigned function of these variables, say $\phi\,(x,\, y,\, z)$, will be real and positive?" When I began to solve the problem I found that, in addition to the particular event whose chance was required, I should have to consider the relations in which this event stood towards several other events on which it more or less depended. It struck me that it would help the memory and facilitate the reasoning, if I registered the various events spoken of in regular numerical order in a table of reference. The event whose chance had to be found resolved itself into a concurrence of two distinct but not independent events 1 and 2; and I denoted the *chance* (or probability) of this concurrence by the symbol $p(1 \bullet 2)$. The compound event $1 \bullet 2$ implied the occurrence of a third event , which it was necessary also to take into account; I had therefore to replace $p(1 \bullet 2)$ by $p(1 \bullet 2 \bullet 3)$. But the consideration of $1 \bullet 2 \bullet 3$ could not be separated from the consideration of a fourth event 4, in conjunction with which it might happen, but not necessarily; and the probability of the concurrence $1 \bullet 2 \bullet 3$ depended materially upon whether it happened in conjunction with this fourth event or without it. Denoting the *non*-occurrence of this fourth event by the symbol : 4, I thus had the equation $p(1 \bullet 2 \bullet 3) = p(1 \bullet 2 \bullet 3 \bullet 4) = p(1 \bullet 2 \bullet 3 : 4)$. Proceeding in this way, but in a somewhat groping and tentative manner, I finally resolved the problem into a form which brought it within the reach of the integral calculus; in other words, *I had somehow determined the limits of integration,* though I hardly knew how. I applied the same method successfully to two or three other problems in the 'Educational Times,' but without being able to make any material improvement in it. Whilst occupied with these researches, the editor of the 'Educational Times' sent me a very neat and simple geometrical solution by Mr. G. S. Carr of the very problem which had given me so much trouble. This so discouraged me (in the belief that I was only wasting my time) that I threw up the whole subject in disgust, and determined for the future to eschew all mathematics that did not fall within the very narrow limits of my requirements as a teacher.

32 When six years afterwards, I broke my resolution and again took up the subject of probability, my mind naturally reverted to the old abandoned method; and it then struck me that, with all its defects, it had one important merit, namely *independence of geometrical diagrams,* and that, consequently, it would be well worth my while to apply myself patiently to the task of removing its defects and developing it, if possible, into something better.

33 My first step was to drop the letter (for *probability*), which I thought might, without ambiguity be left understood; so that, for instance, $1 \bullet 2 \bullet 3$ should replace $p(1 \bullet 2 \bullet 3)$ as an abbreviation for "the probability of the event $1 \bullet 2 \bullet 3$." My next step was to use letters instead of numbers, as A B C instead of $1 \bullet 2 \bullet 3$, and an accent to denote non-occurrence, as ABCinstead of $1 \bullet 2 \bullet 3$. But at this point a difficulty presented itself: how

was A B C, the chance of the compound event A B C, to be distinguished from A B C, the product of the chances A, B, C? for the chance of the compound event would not generally be the product of the chances of the separate events. To guard against this risk of confusion I decided to use capitals, as A B C, when the chance of the whole compound event was meant, and small italic letters, *a*, *b*, *c*, when the product of the separate chances *a*, *b*, *c* was meant. Thus, though the chances A, B, C were separately equal to *a*, *b*, *c*, the symbol A B C would not (except in the case of independent events) be equivalent to the symbol *a*, *b*, *c*. The equivalence of A (B + C) and A B + A C, and of similar expressions, I discovered before I introduced letters instead of numbers.

34 So far, I had made no real advance: the substitution of a literal for a numerical notation improved perhaps the *appearance* of the method; but it did not affect it practically. In applying the method to such questions as required the integral calculus it still remained a tentative method; I still groped my way towards conclusions in particular cases without the help of any general rules of procedure. At last I was struck by the fact that the events registered in my tables, and whose chances were denoted by the letters, were all of the form $x > x_1$, $x_2 > x$, $x_2 > x_1$, $y > y_1$, $y_2 > y$, $y_2 > y_1$, &c., and *had all reference to the limits of the different variables.* This suggested the idea of a partial return to the original numerical notation and classifying the events according to the variable spoken of. I denoted the event and also the chance of the event $x > x_1$ by x_1, the event $x_1 > x$ and its chance by x_1, and so on for x_2, $x_{2'}$, x_3, $x_{3'}$, , y_1, $y_{1'}$, &c. This was a very important step so far as my method related to integration limits; and after this its development in this direction was comparatively rapid—too much so for me to remember very accurately its different stages. Still, I looked upon the method as essentially and inseparably connected with probability; and even when I had decided that it would be more convenient and less confusing to let my symbols denote *logical statements* rather than *mathematical chances,* I could not for some time turn to any account the independence of mathematics which I had thus secured for the method. The notion of the mutual exclusiveness of events (or statements) connected by the sign clung to the method to a very late period; in fact, I was in the very act of writing my first article "On Symbolical Reasoning" for the 'Educational Times,' when the needlessness of this restriction occurred to me. I had written down my definitions of the equations A B C = 1 and A B C = 0 in the following words: —

> The equation A B C = 1 asserts that all three statements are true;
> the equation A B C = 0 asserts that all the three statements are *not* true, *i.e.* that at least *one* of the three is false;

35 and I had to consider suitable definitions of the equations A + B + C = 1 and A + B + C = 0. It was quite evident that the equation A + B + C = 0, whether the statements A, B, C were mutually exclusive or not, must assert that *all the three statements are false*; and *the very words used* in the previous definitions of A B C = 1 and A B C = 0 suggested that, as a symmetrical complement of this, the equation A + B + C = 1 should assert that all the three statements are *not* false, *i.e.* that at least *one* of the three is true. The only question to decide was whether the rule of multiplication,(A + B) (C + D) = A C + A D + B C + B D, would still hold good. A very little consideration showed that it would; so, though the method was correct, so far as it went, on either supposition, I judged it wiser to leave room for possible future development by adopting the wider rather than the narrower hypothesis for its basis.

36 Finding myself thus, at the end of my investigation, on logical instead of mathematical ground, I naturally began to study the relation in which my method stood towards the

ordinary logic, and especially towards the syllogism. The only book on logic that I possessed was Prof. Bain's work; and to this I turned. The resemblance which my method bore to Boole's, as therein described, of course struck me at once; but Boole's treatment of the syllogism was more likely to put me on the wrong track than to help me. As my most elementary symbols denoted *statements,* not necessarily connected with quantity at all, I could not see how the syllogism, with its ever recurring *all, some, none,* could be brought within the reach of my method. The Cartesian system of analytical geometry at last supplied the desiderated hint as to the proper mode of procedure. In this system, as every mathematician knows, *one single point* is spoken of in every equation, but with the understanding that it is a *representative* point, and that the equational statement made respecting it is also true respecting every other point in the locus expressed bythis equational statement.

37 The symbol :, which I had already begun to use as an occasionally convenient abbreviation for the word "implies," now became almost imperative. Syllogistic reasoning is strictly restricted to *classification.* The statement "All X is Y" is equivalent to the conditional statement "If any thing belongs to the class X, it must also belong to the class Y." Speaking then of something originally unclassed, if x denote the statement "It belongs to the class X," and if y denote the statement "It belongs to the class Y," then the implicational statement $x : y$ (or x implies y) will be equivalent to the syllogistic statement "All X is Y."

38 It was evident after this that $x : y'$ would be the proper symbolical expression for "No X is Y;" but, strange to say, the discovery of the suitable symbolic expressions for "Some X is Y " and "some X is not Y " caused me no small trouble, even though I had previously more than once wondered under what circumstances the symbol $(x : y)'$ would be required. For a long time I did not recognize this $(x : y)'$ as the equivalent (in classi[fi]cation) of "Some X is not Y," and $(x : y')'$ as the equivalent of "Some X is Y." In my second communication to the Mathematical Society I used the symbol $v : xy$ to denote "Some X is Y;" and it was only when I had read the very just objection made by one of the referees to my introduction of the arbitrary and possibly non-existent class V that it suddenly flashed upon me that the true symbolical expression for "Some X is Y " should be $(x : y')'$, the denial of the implication $x : y'$, and that the true symbolical expression for "Some X is not Y " should be $(x : y)'$, the denial of $x : y$.

39 The next new symbol to be introduced into my symbolic system was the symbol x_a, to express the *chance of x being true on the assumption that a is true.* The circumstances which suggested this symbol to me are curious and instructive. My first idea was to use the symbol x_c to denote the chance of x being true, the suffix c being merely suggestive of the word *chance* and not denoting a statement. In fact, this was the notation which originally formed the basis of my fourth paper, "On the Calculus of Equivalent Statements," when it was first communicated to the London Mathematical Society. While this paper was in the hands of the referees, I was occupied with a problem proposed to me by Mr. C. J. Monro, and involving among other things the consideration of a chance $(xz)_c$, which I at first considered as equal to $x_c\,(x : z)_c$, being under the idea that, since $x : z$ expressed the conditional statement "If x is true z is true," $(x : z)_c$ would be the proper symbol to express the *chance* that if x is true z is true. On reflexion I discovered that this, plausible as it sounded, would lead to inconsistency of notation. For, since $x : z$ is equivalent to $z' : x'$, consistency of notation required that $(x : z)_c$ should denote the same chance as $(z' : x')_c$; and, *as I had interpreted the symbols,* this would not be the case. The chance that z is true,

on the assumption that is true, is not generally equal to the chance that x is false on the assumption that z is false. I was thus forced to the conclusion that I had put a wrong interpretation on the symbols $(x:z)_c$ and $(z':x')_c$, which *must* be equivalent, and that neither of them, therefore, was the proper symbol for the chance which I wished to express. It became necessary, therefore, since there did not appear to be sufficient data for logically *inferring* a correct expression for this chance, to invent a new and arbitrary symbol for it; and then the important question presented itself as to what that symbol should be. It must, if possible, be brief and easily formed; it must be formed, at least partly, of the symbols and z; and yet it must be some *unambiguous* combination of those symbols—that is to say, a combination which should convey no other meaning either by definition or by implication. One of several symbols that offered themselves as candidates for the important post to be filled, I at last selected the symbol z_x as the one most likely to perform effectively the duties required of it.

40 The symbol z_x being thus fairly installed, I was struck by the resemblance between it in some respects and the symbol z_c. Both expressed the *chance* of the truth of z, though on generally different assumptions; and, what was more remarkable, some of the formulae which I had obtained involving the constant suffix c, as

$$(\alpha + \beta)_c = \alpha_c + \beta_c - (\alpha\beta)_c ,$$

were also true when for c I substituted the variable suffix x. This suggested the propriety of considering c too as a *statement,* instead of a mere arbitrary abbreviation for "the chance of the truth of," and it soon became evident what that statement must be. The constant suffix c, like the variable suffix x, must denote *a statement taken for granted*; but, unlike the variable x, it must denote a statement whose truth is taken for granted *always* —that is, *throughout the whole of an investigation.* In other words, the suffix c must be *an exact equivalent for the logical symbol* 1.

41 This, however, necessitated other symbolical changes. As long as the *suffixc* did not denote a *statement,* I was at liberty to use this letter in conjunction with the letters and in *other positions* as a *statement*; so that c_c (like a_c and b_c) would simply denote the chance of the truth of the statement c, and its value might vary from to ; but with the new meaning of the suffix c, we should always have $c_c = 1$. It thus became expedient to leave c at liberty to discharge other functions in company with its old comrades a and b, and to intrust the duty of denoting universally admitted statements to some letter whose services in other capacities could be more easily spared. I decided, after some hesitation, on the Greek letter ε, which is easily formed, pleasing to the eye, and not often wanted. It may be asked, why was I not satisfied with the symbol , which already denoted an admitted statement? My answer is, first, that I thought this numeral would not *look well* in frequent companionship with *literal* suffixes; and, next, that I thought it better to reserve it, in company with other numerals, for distinguishing statements of the same class or series, as a_1, a_2, a_3 &c., which, though different statements, will generally be found to have some common factor or characteristic a.

42 Having thus decided that ε should denote a statement of acknowledged truth, that x_a should denote the chance of the truth of x on the assumption that a is true, and that therefore x_ε must simply denote the chance that x is true, *with no assumption beyond the understood data of the problem,* it soon became evident that this notation would express many of the laws of probability in neat and compact formulae, and also that it would contribute towards precision of reasoning from its constant reference, by means of its

suffixes, to the assumptions on which any argument in probability rested. It is well known that of all mathematical subjects probability is the one in which mistakes are most apt to be made; and these mistakes are usually the result of correct reasoning based upon unperceived false assumptions. These assumptions, for the most part, would be readily seen to be false, if they were only *expressed*; a notation therefore that actually forces them on the attention must be considered as possessing one very important advantagein that fact alone.

43 As this new scheme of probability-notation quite superseded that which formed the basis of the paper which had been already submitted to the referees of the Mathematical Society, these gentlemen naturally declined (on the scheme being communicated to them through the Honorary Secretary, Mr. Tucker) to pronounce any opinion either upon the original paper or on the proposed alterations, till the whole was recast and rewritten. When this was done, and the paper again submitted to them, they advised its publication.

44 In my former paper in 'Mind,' "On Symbolical Reasoning," I referred to the analogy between the relation connecting *antecedent* and *consequent* in logic and that connecting *subject* and *predicate* in grammar. Would it be presumptuous to suggest as a probable hypothesis that this analogy is more than a mere coincidence, and that it really points to an original identity? It does not seem unreasonable to suppose that in the very early stage of human speech, each separate word represented a *complete statement* and conveyed its own independent information. On this supposition, the growth of vocal language would proceed according to laws in some respects analogous to those which shape the development of a language of symbols. Our abstract nouns, for instance, seem to be nothing but abbreviations for original statements. Take the compact and well-known saying, "Unity is strength." What is this but an abbreviation for the conditional statement, "If a company be *united,* they will be *strong*"? or, as it may be otherwise expressed, "If the statement symbolized by the abbreviation *unity* for 'They are united' be applicable to a company of persons, so will also the statement symbolized by the abbreviation *strength* for 'They are strong.' "

45 But here I must stop. Speculations as to the primaeval forms of human speech do not come fairly within the limits prescribed by the title of this article; and further discussion of the subject in this direction would therefore be irrelevant.

NOTES

†. To avoid the employment of brackets and repetition as much as possible, itwill be convenient to use *both*, with this distinction, that the symbols : and ∷ should be coordinate (i.e. of *equal reach* in regard to the statements affected by them), but both subordinate (i.e. of *inferior reach*) to the symbol =.

Thus, $a : b + c :: d + e : f :: g$is an abbreviation for the complex statement

$(a : b + c)(b + c :: d + e)(d + e : f)(f :: g)$

while $a : b + c = d + c : f :: g$ is an abbreviation for

$(a : b + c) = (d + c : f :: g)$

‡. The subject of probability was one which I had recently taken up at the request of Mr. J. C. Miller, the mathematical editor of the 'Educational Times,' who felt great interest in it himself, and strongly recommended it as "an unworked vein in which I should find many treasures."

*. That the conceptions which underlie the very elements of symbolical mathematics are by no means easy to grasp will be admitted by any one who has attempted to explain to beginners the real logical meaning of the "Rule of Signs" in multiplication and division of ordinary algebra. This difficulty meets the tyro on the very threshold of the science. When he has advanced a few steps further, he is confronted with a symbolical paradox which it has taxed the ingenuity of the subtlest mathematical intellects to explain, and of which no rational explanation whatever was given until within very recent times; I allude to the useful and important yet perplexing symbol $\sqrt{1}$.

ABSTRACTS

This paper discusses in a general way the cases in which symbols may be advantageously employed in logical and mathematical reasoning, and endeavours, from an examination of our existing stock of symbols, to deduce some rules or guiding principles, first, as to when new symbols should be introduced, and secondly, as to what kinds of symbols should be selected. It also contains a brief account of the gradual development of the author's own symbolic method, as explained and set forth in his papers published in the *Proceedings of the London Mathematical Society*, the *Educational Times*, the *Philosophical Magazine*, and *Mind*.

[Résumé: Cet article discute de façon générale les cas dans lesquels les symboles peuvent être utilisés de façon avantageuse dans le raisonnement logique et mathématique, et aspire, par un examen des symboles déjà existants, à déterminer quelques règles ou principes, d'abord, quant à l'opportunité d'introduire de nouveaux symboles, puis, quant au types de symboles qui doivent être sélectionnés. Cet article contient aussi une brève histoire du développement graduel de la méthode symbolique propre à l'auteur, telle qu'elle a été présentée et expliquée par l'auteur dans ses articles publiés dans les *Proceedings of the London Mathematical Society,* le *Educational Times*, le *Philosophical Magazine,* et *Mind*.]